DEPARTMENT OF MATHEMATICS

CALCULUS

B.Sc., MATHEMATICS/CMA11
E CONTENT

CALCULUS

Course Objectives

1. This course introduces the basic concepts of differential and integral calculus.

2. To know about the angle between two curves and the radius of curvature.

3. To inculcate a strong knowledge about evolutes and envelopes.

4. Knowledge about reduction formulae and properties of definite integrals.

5. To acquire the knowledge about evaluation of double and triple integrals.

Course Outcomes

1. After studied unit -1, the student will be able to determine the extreme values of the given function. 2. After studied unit -2, the student will be able to demonstrate knowledge of Cartesian and polar coordinates.

3. After studied unit -3, the student will be able to gain knowledge of curvature, evolutes, and envelope concepts.

4. After studied unit -4, the student will be able to evaluate definite integration problems and able to apply reduction formulae.

5. After studied unit -5, the student will be able to evaluate double and triple integrals.

syllabus

UNIT-I: Differential Calculus

n th derivative - Leibnitz‟s theorem (Without proof) and its application - Jacobians - Total differential - Maxima and Minima functions of two and three independent variables - Lagrange‟s method (without proof) - Simple problems.

UNIT-II: Differential Calculus (Contd…)

Polar coordinates – Relation between Cartesian and Polar coordinates - Polar Equation of a Straight line, Circle and Conic only (Related problems not necessary) - Angle between radius vector and tangent – Angle between two curves – Curvature - Radius of Curvature in Cartesian and Polar coordinates.

UNIT-III: Differential Calculus (Contd…)

Centre of Curvature – Evolutes – Envelopes – Asymptotes – Methods of finding asymptotes (Rational algebraic curves only)

UNIT-IV: Integral Calculus

Reduction formula for sinn x, cos n x, tann x, sinm xcos n x - Beta and Gamma Functions - Properties and Problems – Definite Integral – Properties - Simple Problems.

UNIT-V: Integral Calculus (Contd…)

Double Integrals - Change of order of Integration - Triple Integrals - Applications to Area, Surface Area and Volume.

Text book: S.Narayanan and T.K.Manicavachagom Pillay (2004) Calculus.S.Viswanathan Printers & Publishers Pvt. Ltd. Chennai.

Reference Books:

1. P.Kandasamy, K.Thilagavathy (2004), Mathematic for B.Sc. Vol.-I, II, III & IV, S.Chand& Company Ltd., New Delhi-55.

2. Shanti Narayan (2001) Differential Calculus. Shyamlal Charitable Trust, New Delhi.

3. Shanti Narayan (2001) Integral Calculus.S.Chand& Co. New Delhi.

4. S.Sudha (1998) Calculus. Emerald Publishers, Chennai.

5. G.B.Thomas and R.L.Finney. (1998) Calculus and Analytic Geometry, Addison Wesley (9thEdn.), Mass. (Indian Print)

6. P.R.Vittal. (2004) Calculus, Margham Publication, Chennai

Successive differentiation – nth derivatives

If y is a function of x , its derivative dy/dx will be some other function of x and the differentiation of this function with respect to x is called second derivative and is denoted by d^2y/dx^2

i.e., $d/dx(dy/dx) = d^2y/dx^2$

Example

1. If $y = \frac{ax+b}{cx+d}$ Find d^2y/dx^2.

Solution:

$$Y = \frac{ax+b}{cx+d}$$

$$\frac{dy}{dx} = \frac{(cx+d)a-(ax+b)c}{(cx+d)^2}$$

$$= \frac{acx+ad-acx-bc}{(cx+d)^2}$$

$$= \frac{ad-bc}{(cx+d)^2}$$

$$\frac{d^2y}{dx^2} = \frac{0-(ad-bc)(2)(cx+d)c}{(cx+d)^4}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{-2c(ad-bc)}{(cx+d)^3}}$$

2. If x=a(cos t + t sin t, Y=a(sin t – t cos t) Find d^2y/dx^2.

Solution:

Y=a(sin t – t cos t)

dy/dt = a(cos t +t sin t – cos t)

= at sin t

x=a(cos t + t sin t)

dx/dt = a(-sin t + t cos t + sin t)

= at cos t

dy/dx = at sint/at cost

=tan t

$\frac{d^2y}{dx^2}$ = d/dt(tan t).dt/dx

= $\sec^2$t.1/at cost

$$\boxed{\frac{d^2y}{dx^2} = \sec^3 t/at}$$

Standard nth derivatives

1. **Nth derivative of e^{ax}**

Solution:

$$y = e^{ax}$$

$$y_1 = e^{ax}.a$$

$$y_2 = e^{ax}.a^2$$

$$y_3 = e^{ax}.a^3$$

$$y_n = e^{ax}.a^n$$

2. **nth derivative of $\frac{1}{ax+b}$**

Solution:

$$y = \frac{1}{ax+b} = (ax+b)^{-1}$$

$$y_1 = (-1)(ax+b)^{-2}.\ a$$

$$y_2 = (-1)(-2)(ax+b)^{-3}.\ a^2$$

$y_3 = (-1)(-2)(-3)(ax+b)^{-4}.\ a^3$

$y_n = (-1)(-2)(-3).....(-n)(ax+b)^{-(n+1)}.\ a^n$

$y_n = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$

Y	y_x
e^{ax}	$a^n e^{ax}$
$\frac{1}{ax+b}$	$\frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$
$\frac{1}{(ax+b)^2}$	$\frac{(-1)^n (n+1)!\, a^n}{(ax+b)^{n+2}}$
Log(ax+b)	$\frac{(-1)^{n-1} (n-1)!\, a^n}{(ax+b)^n}$
Sin(ax+b)	$a^n \sin(ax+b+n\pi/2)$
Cos(ax+b)	$a^n \cos(ax+b+n\pi/2)$

Example

1. find the nth derivative of $\frac{2x+1}{(2x-1)(2x+3)}$

Solution:

$$\text{let } y = \frac{2x+1}{(2x-1)(2x+3)}$$

$$\text{let } \frac{2x+1}{(2x-1)(2x+3)} = \frac{A}{(2x-1)} = \frac{B}{(2x+3)}$$

$$2x+1 = A(2x+3) + B(2x-1)$$

Put $x = ½$

$$2 = A(4)$$

$A= ½$

Put $x = -3/2$

$-2 = B(-4)$

$B = ½$

$$Y = \frac{1/2}{(2x-1)} + \frac{1/2}{(2x+3)}$$

$$\boxed{Y_n = (-1)^n 2^{n-1} n! \left[\frac{1}{(2x-1)^{n+1}} + \frac{1}{(2x+3)^{n+1}}\right]}$$

LEIBNIT'Z THEOREM

If u and v are functions of x and n is a positive integer then

$$\mathbf{D^n(uv) = u_n v + {}^nc_1 u_{n-1} v_1 + {}^nc_2 u_{n-2} v_2 + \dots\dots\dots + {}^nc_r u_{n-r} v_r + \dots\dots\dots + u v_n}$$

Where $D^n(uv)$ standards for the nth derivative of uv

Example

1. find the nth derivative of x^2e^{5x}

Solution:

$u = e^{5x}$ and $v = x^2$

then $D^n(uv) = u_n v + {}^nc_1 u_{n-1} v_1 + {}^nc_2 u_{n-2} v_2 + \dots\dots\dots + {}^nc_r u_{n-r} v_r + \dots\dots\dots + u v_n$

$$= 5^n e^{5x} x^2 + {}^nc_1 5^{n-1} e^{5x}.2x + {}^nc_2 5^{n-2} e^{5x}.2$$

$$= e^{5x}.5^{n-2}\left[5^2x^2 + (n)(5)(2x) + \frac{n(n-1)}{2}.2\right]$$

$$\boxed{D^n(uv) = e^{5x}.5^{n-2}[25x^2 + 10nx + n(n-1)]}$$

JACOBIAN METHOD:

In this section, we explore the concept of a "derivative" of a coordinate transformation, which is known as the Jacobian of the transformation. However, in this course, it is the determinant of the Jacobian that will be used most frequently.

If we let u = (u , v); p = (p , q) and x = (x , y) then (x; y) = T (u; v) is given in vector notation by x = T (u)

This notation allows us to extend the concept of a total derivative to the total derivative of a coordinate transformation.

In non-vector notation, the total derivative at a point (p; q) of a coordinate transformation T (u; v) is a matrix J (u; v) evaluated at

(p; q) it can be shown that this matrix is given by

$$J(u; v) = \begin{matrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{matrix}$$

Example:

Calculate the Jacobian Determinant of T (u; v) = (u^2-v, u^2+v)

Solution:

If we identify x = u^2-v and y = u^2+v

$$\partial (x; y) \div \partial (u; v) = \frac{\partial x \partial y}{\partial u \partial v} - \frac{\partial x \partial y}{\partial v \partial v}$$

$$= (2u)(1) - (-1)(2u)$$

$$\boxed{\partial (x; y) \div \partial (u; v) = 4u}$$

Lagrange Multipliers Theorem:

The mathematical statement of the Lagrange Multipliers theorem is given below.

Suppose $f : R^n \to R$ is an objective function and $g : R^n \to R$ is the constraints function such that $f, g \in C^1$, contains a continuous first derivative. Also, consider a solution x^* to the given optimization problem so that $\text{ran}Dg(x^*) = c$ which is less than n.

Objective function: Max f(x)

Constraints: Subject to

$g(x) = 0$,

where $Dg(x^*)$ = Matrix of partial derivatives, i.e., $[\partial g_j/\partial x_k]$, then there exists a unique Lagrange multiplier λ^* in R^c so that $Df(x^*) = \lambda^{*T}Dg(x^*)$

Let's understand how to define the method of Lagrange multipliers for both single and multiple constraints so that we can easily solve many problems in mathematics.

Lagrange Multiplier Theorem for Single Constraint

In this case, we consider the functions of two variables. That means the optimization problem is given by:

Max f(x, Y)

Subject to:

$g(x, y) = 0$ (or)

We can write this constraint by adding an additive constant such as g(x, y) = k.

Let us assume that the functions f and g (defined above) contain first-order partial derivatives. Thus, we can write the Lagrange function as:

$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda\, g(x, y)$

If $f(x_0, y)$ is the maximum point of the function f(x, y) for the given constrained problem and $\nabla g(x_0, y_0) \neq 0$ then there will be λ_0 so that (x_0, y_0, λ_0) is known as a stationary point for the above Lagrange function.

However, we can find the maximum and minimum values for the function f(x, y) subject to the constraint g(x, y) = 0 using the below equations involving partial derivatives.

$(\partial f/\partial x) + \lambda(\partial g/\partial x) = 0$

$(\partial f/\partial y) + \lambda(\partial g/\partial y) = 0$

$g(x, y) = 0$

Also, it is not mandatory to estimate the explicit values for Lagrange multiplier λ.

Similarly, for the functions of f and g in three variables, say x, y, and z, we use the following equations to determine the maximum and minimum values of f. They are:

$(\partial f/\partial x) + \lambda(\partial g/\partial x) = 0$

$(\partial f/\partial y) + \lambda(\partial g/\partial y) = 0$

$(\partial f/\partial z) + \lambda(\partial g/\partial z) = 0 \qquad , g(x, y, z) = 0$

MAXIMUM AND MINIMUM:

Definition:

Let f (x) be a real function defined on an interval [a,b].Then, f(x) is said to have the maximum value in [a, b], if there exist a point c in [a, b] such that f(x) $\leq$ f(c) for all x ϵ [a, b] Minimum - Let f (x) be a real function defined on an interval [a, b]. Then, f(x) is said to have the minimum value in [a, b], if there exist a point c in [a, b] such that f(x) = f(c) for all x ϵ [a, b].

1. Find the maximum and minimum values , if any , without using derivatives of the following functions:

(A)$f(x) = 4x^2 - 4x + 4$ on R

(B) $f(x) = (x-1)^2+2$ on R

Solution - We have $f(x) = 4x^2-4x+4$

$= (2x-1)^2+3$ Clearly, $(2x-1) \succ 0$ for all XER

$(2x-1)^2+3 \geq 3$

$= f(x) \geq f(\frac{1}{2})$

Hence, the function f(x) = 4x2 - 4x + 4 defined on R is minimum and the minimum value is 3 for $x = \frac{1}{2}$

Solution -

We have $f(x) = -(x-1)^2+2$ Clearly, (x-1)2$\geq$0 for all xϵ R

$=- (x-1)^2 \leq 0 \qquad = \ - (x-1)^2+2 \leq 2$

= f(x) ≤ f(1)

Hence, the function f(x) = (x-1)2 + 2 defined on R is maximum and at x= 1 the maximum value is 2.

LOCAL MAXIMA AND LOCAL MINIMA:

Definition:

1. Local Maxima -- A function f(x) is said to attain a local maximum at x = a if

there exist a neighborhood (a-h, a+ h) such that

f(x) < f(a) for all x ϵ (a-h,a+ h), x ≠ a

2. Local Minima - A function f(x) is said to attain a local minimum at x = a if there exist a neighborhood (ah, a + h) such that

f(x)> f(a) for all x (a-h, a+h),x ≠a

FIRST DERIVATIVE TEST FOR LOCAL MAXIMA AND LOCAL MINIMA:

Let f(x) be a function

1. Put y = f(x)

2. Find $\frac{dy}{dx}$

3. Put $\frac{dy}{dx}$= 0 and find the roots of x (say x1, x2 and so on) dx

4. Make an interval around x, such that (x_1 -h, x_1+ h)

5. Putting x-h and x, + h in $\frac{dy}{dx}$

6. If it changes from positive value to negative value, then local maxima and if it is from negative to positive then local minima.

7. If does not change sign then neither maxima nor minima at the point x = x1

Find the points of local maxima or local minima for the following functions, using the first derivative test:

Example:

1. $f(x) = x^3 - 6x^2+9x+15$

Solution:

Let $f(x)=x-6x^2+9x+15$

Differentiate with respect to x

$f'(x) = 3x^2 - 12x + 9$

Let $f'(x)=0$

$3x^2-12x+9=0$

$x^2-4x+3=0$

$x=3,1$

Case-1:

For x = 3 let us consider an open interval around 3 such that (2.9 , 3.1)

Putting it in f(x), we get $f'(x) =3x^2-12x+9$

$f'(2.9)=3(2.9)2 - 12(2.9) + 9 = 25.23-34.8+9=-0.5$

$f'(3.1)=3(3.1)^2 - 12(3.1) + 9 = 28.83-37.2+9= 0.63$

Since it changes sign from negative to positive so the function is minimum at x=3

So, x = 3 is the point of local minimum.

Case-2:

For x=1 let us consider an open interval around 1 such that

(0.9,1.1)

Putting it in f'(x),we get

$f'(x) = 3x^2 - 12x+9$

f'(0.9)=3(0.9)212(0.9) +9 = 2.43 10.8+ 9 = 0.63

f'(1.1)=3(1.1)2-12(1.1) +9 = 3.6313.2+9= -0.57

Since it changes sign from positive to negative so the function is maximum

at x = 1

So, x = 1 is the point of local maximum.

ASYMPTOTES:

A Straight line is said to be asymptotes to a curve if as the point p moves along the curve in perpendicular distance from p on the Straight line tends to zero.

EXAMPLE:

We See that the perpendicular distance PM from a point p on the RH $xy=c^2$ tends to zero as y tends to ∞ Therefore, y-axis is an asymptote to the RH $xy=c^2$. Similarly x-axis is also an asymptote to RH $xy=c^2$

GENERAL METHOD OF FINDING ASYMPTOTES OF ALGEBRAIC CURVE:

Let f(x,y)=0 be an algebraic curve of degree n. Let this equation be written in the form,

$(a_0x^n+a_1x^{n-1}y+a_2x^{n-2}y^2+.............+a_{n-1}xy^{n-1}+a_ny^n)+(b_0x^{n-1}+b_1x^{n-2}y+b_2x^{n-3}y^2+.............+b_{n-1}xy^{n-1})+(c_0x^{n-2}+c_1x^{n-2}y+.............+c_{n-2}xy^{n-2})+..........+k=0$

This equation can also be written in the form

$x^n\phi n(\frac{y}{x}) + x^{n-1}\phi n-1(\frac{y}{x}) + x^{n-2}\phi n-2(\frac{y}{x}) +........\phi(\frac{y}{x})=0$

WORKING RULE FOR FINDING ASYMPTOTES:

1. Put x=1 & y-m in the highest degree(say n) terms of f(x,y) =0 & hence write down ϕ(m)=0.
2. Solve ϕ(m) = 0 for m & determine the real of m say,(m_1, m_2,, m_3)
3. Consider the next lowest degree (m-1) terms of f(x,y)=0, put x=1,y=m & determine $\phi n-1(m)$.
4. The value of c of the asymptote y=mx+c corresponding to the values of m are given by $c= -\frac{\phi n-1(m)}{\phi' n(m)}$ if $\phi n'(m) \neq 0$
5. If $\phi' n$ (m) =0 for any values of m & $\phi n-1$(m)≠ 0 then the value is no asymptote Corresponding to this value of m.

PROBLEMS:

1. Find the asymptotes $3x^2 + 2x^2y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5y=0$

solution:

The given eq is $3x^2 + 2x^2y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5y=0$

The highest degree (n=3) terms are $3x^2 + 2x^2y - 7xy^2 +2y^3 = 0$

Put x=1, y=m

$\phi n\ (m) \Rightarrow \phi 3\ (m) = 3 + 2m + 7m^2 + 2m^3$

$\phi 3\ (m) = 2m^3 - 7m^2 + 2m + 3$

Put $\phi 3(m) = 0$

$2m^3 - 7m^2 + 2m + 3$

1	2	-7	2	3
	0	2	-5	-3
	2	-5	-3	0

$2m^2 - 5m - 3=0$

$m=-\frac{-1}{2}$, m=3 and m=1

Here the value of m are different, then to find c

$$c= - \frac{\phi n-1(m)}{\phi' n(m)} \Rightarrow c= - \frac{\phi 2(m)}{\phi' 3(m)}$$

Here the next highest degree (n-1=2) of the terms

$\phi 2(m)$ = -14xy + 7y^2

Put x=1 , y =m

$\phi 2(m)$ = -14m + 7m^2

$\phi' 3(m)$ = 6m^2 - 14m + 2

c=-$\frac{[-14m+7m2]}{6m2-14m+2}$

Put m=1

c=- $\frac{[-14+7]}{6-14+2} \Rightarrow \frac{14-7}{-6} \Rightarrow \frac{-7}{6}$

c= $\frac{-7}{6}$

Put m=$\frac{-1}{2}$

c= - $\frac{\left[-14(\frac{-1}{2})+7(\frac{-1}{2})\right]}{6(\frac{-1}{2})2-14(\frac{-1}{2})+2}$

= $\frac{-35}{7} \times \frac{2}{21}$

c=$\frac{-5}{6}$

Put m=3

c= $\frac{-[4(3)+7(3)2]}{6(3)2-14(3)+2}$

c=$\frac{-21}{14}$

$c=\frac{-3}{2}$

$y= mx+c$

When $m=1$ and $c=\frac{-7}{6}$

$y=x - \frac{7}{6}$

$y= \frac{6x-7}{6}$

$6y-6x+7=0$

When $m=\frac{-1}{2}$ and $c=\frac{-5}{6}$

$y= \frac{-1}{2}x- \frac{5}{6}$

$3x+6y+5=0$

When $m=3$ and $c=\frac{-3}{2}$

$y=3x- \frac{3}{2}$

$6x - 2y -3=0.$

2) Find the asymptotes of $x^3 + y^3 = 3axy$

Solution:

The given equation is $x^3 + y^3 = 3\,axy$

$x^3 + y^3 - 3\,axy = 0$

The highest degree (n=3) terms are $x^3 + y^3 = 0$

Put x=1, y=m

$1^3 + m^3 = 0$

$1^3 + m^3 = (m+1)(m^2 - m + 1)$

$(m+1)(m^2 - m + 1) = 0$

$m = -1$, $m = [-b \pm \sqrt{(b^2 - 4ac)}]/2a$

$m = \frac{1 \pm \sqrt{-3}}{2}$

$m = \frac{1 \pm \sqrt{i3}}{2}$

It is imaginary put m=-1 roots vomit value of m

Put m= -1

$c = -\frac{\phi n-1(m)}{\phi\prime n(m)}$

$\Rightarrow c = -\frac{\phi 2(m)}{\phi\prime 3(m)}$

$\phi(m) = 1^3 + m^3 = 0$

$\phi'3(\mathrm{m})=3\mathrm{m}2$

Here the next highest degree (n-1=2) of the terms $\phi 2(m)$ = -3axy

Put x=1 , y=m

$\phi 2(m)$= -3am

c= a/m

Put m= -1

c= -a

y= mx +c

y= -x +c

x+y+a=0

Methods of finding asymptotes partial to x-axis :

1. To find asymptotes parallel to x-axis equal to the coefficient of highest degree x to 0
2. To find asymptotes parallel to y-axis equal to the coefficient of highest degree x to 0.

PROBLEMS:

1. Find the asymptotes to the curve x^2y-y-x=0

Solution:

The equation of curve x^2y-y-z=0

The equation of the asymptote paralled to the x-axis are got by the equating of the coefficient of highest powers x to 0

y=0 is the only asymptote parallel to x-axis

Similarly, the asymptotes parallel to y-axis is got by equating the coefficient of y to 0

Is that x2-1 =0

(x-1)(x-2)=0 are asymptote parallel to x-axis

Hence the asymptotes x-1, x+1=0, y=0.

ENVELOPE

A curve which touches each member of a given family of curves is called envelope of that family.

Procedure to find envelope for the given family of curves:

Case 1: Envelope of one parameter family of curves Let us consider $y = f(x,\alpha)$ to be the given family of curves with 'α' as the parameter.

Step 1: Differentiate w.r.t to the parameter α partially, and find the value of the parameter

Step 2: By Substituting the value of parameter α in the given family of curves, we get the required envelope.

Special Case: If the given equation of curve is quadratic in terms of parameter,

i.e. Aα2+Bα+c=0, then envelope is given by **discriminant = 0 i.e. B2 - 4AC=0**

Case 2: Envelope of two parameter family of curves.

Let us consider $y = f(x,\alpha, \beta)$ to be the given family of curves, and a relation connecting the two parameters α and β, $g(\alpha, \beta) = 0$

Step 1: Consider α as independent variable and β depends α .

Differentiate $y = f(x,\alpha, \beta)$ and $g(\alpha, \beta) = 0$, w.r. to the parameter α partially.

Step 2: Eliminating the parameters α, β from the equations resulting from step 1 and

$g(\alpha, \beta) = 0$,

we get the required envelope.

Problem

Determine the envelope of $x\sin\theta - y\cos\theta = a\theta$, where θ being the parameter.

Solution :

Differentiate , $x\sin\theta - y\cos\theta = a\theta$ (1)

with respect to θ ,

we get, $x\cos\theta + y\sin\theta = a$ (2)

As θ cannot be eliminated between (1) and (2) , we solve (1) and (2) for x and y in terms of θ.

For this, multiply (2) by $\sin\theta$ and (1) by $\cos\theta$ and then subtracting,

we get, $y = a(\sin\theta - \theta\cos\theta)$. Using similar simplification, we get, $x = a(\theta\sin\theta + \cos\theta)$.

Envelope of Two parameter family of curves :

problem

Find the envelope of family of straight lines ax+by=1, where a and b are parametersconnected by the relation ab = 1.

Solution :

$ax + by = 1$ (1)

$ab = 1$ (2)

Differentiating (1) with respect to a (considering ‘a’ as independent variable and ‘b’

depend on a).

$X + \frac{db}{da} y = 0$

i.e $\frac{db}{da} = -\frac{x}{y}$ (3)

Differentiating (2) with respect to a

$b + a\frac{db}{da} = 0$

i.e $\frac{db}{da} = -\frac{b}{a}$ (4)

From (3) and (4), we have

$\frac{x}{y} = \frac{b}{a}$

i.e. $\frac{ax}{1} = \frac{by}{1} = \frac{ax+by}{2} = \frac{1}{2}$

$\therefore$ $a = \frac{1}{2x}$ and $b = \frac{1}{2y}$ (5)

Using (5) in (2), we get the envelope as 4xy = 1.

Problems on Evolute as envelope of its normals :

Determine the evolute of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2}$ =1 by considering it as an envelope of its normal.

Solution :

Let P (a cosht, b sinht) be any point on the given hyperbola. Then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{bcosht}{asinht} = \frac{b}{a}\,\text{cotht}$$

Equation of normal line to the hyperbola is

$$(\text{y-bsinht}) = -\frac{a}{bcosht}(\text{x-acosht}) \qquad (1)$$

$$\Rightarrow \frac{by}{sinht} + \frac{ax}{cosht} = a^2 + b^2 \qquad (2)$$

Differentiating (2) partially with respect to t, we have,

$$\frac{-by}{(sinht)^2}\,\text{cosht} - \frac{-by}{(sinht)^2}$$

Method of finding asymptotes by inspection method

If the equation of the curve can be expressed in the form $P_n + P_{n-2} = 0$. Where P_n contains n linear factors . Then the asymptotes of the curve is given by $P_n = 0$

1.Find all the asymptotes of $x^2/a^2 – y^2/b^2 = 1$

Solution:

Given equation is; $x^2/a^2 – y^2/b^2 = 1$

$X^2/a^2 – y^2/b^2 – 1 = 0$

This equation of the form $P_n + P_{n-2}=0$

The asymptotes are given by, $P_n = 0$

$X^2/a^2 – y^2/b^2 = 0$

$(x/a – y/b)\ (x/a + y/b) = 0$

$\therefore$ The asymptotes are $(x/a – y/b)=0$ &$(x/a + y/b) = 0$.

2. Find the asymptotes of (x+y)(x-y)(x-2y-4) = 3x+7y-6

Solution:

The given equation is ; (x+y)(x-y)(x-2y-4) = 3x+7y-6

[(x+y)(x-y)(x-2y-4)] – [3x+7y-6] = 0

This equation of the form $P_n + P_{n-2}=0$

$P_n = 0$, (i.e.,) (x+y)(x-y)(x-2y-4) = 0

$\therefore$The asymptotes are x+y = 0, x-y = 0, x-2y-4 = 0

3. Find the asymptotes of the curve y3-6xy2+11x2y-6x3+x+y = 0

Solution:

Given equation is , y3-6xy2+11x2y-6x3+x+y = 0

It is in the form Pn + Pn-2=0

Pn = y3-6xy2+11x2y-6x3

Pn-2 = x+y

Pn =0,

The asymptotes are given by ,

Y3-6xy2+11x2y-6x3

(y-x)(y-2x)(y-3x)=0

...The asymptotes are y-x=0, y-2x =0, y-3x=0.

4. Show that the 8 point of the intersection of the curve $x^4-5x^2y^2+4y^4+x^2-Y^2+x+y+1=0$ and it is asymptotes lie on rectangular hyperbola.

Solution:

Given equation is $x^4-5x^2y^2+4y^4+x^2-Y^2+x+y+1$

This can be written as,

$X^4-x^2y^2-4x^2-Y^2+4y^4=0$

$X^2(x^2-y^2) – 4y^2 (x^2-y^2) =0$

$(x^2-4y^2) (x^2-y^2)=0$

This equation of the form $P_n + P_{n-2}=0$

Where, $P_n = (x^2-4y^2)(x^2-y^2)$

$P_{n-2} = x^2-y^2+x+y+1$

(i.e.,) $x^4-5x^2y^2+4y^4+x^2-Y^2+x+y+1=0$

∴ The asymptotes are given by $P_n=0$

$(x^2-4y^2) (x^2-y^2)=0$

$X^2-4y^2=0$ ---------> ①

$X^2-y^2=0$ ---------> ②

①=> $x^2-(2y)^2=0$

$(x+2y)(x-2y)=0$

②=>$(x^2-y^2)=0$

$(x-y)(x+y)=0$

∴The asymptotes are , $x+2y=0$, $x-2y=0$, $x-y=0$, $x+y=0$

There are the point of intersection of the asymptotes of a curve and the asymptotes lie on the curve, $P_{n-2}=0$.

(i.e.,) $x^2-y^2+x+y+1=0$

Since , $P_n=0$ is of degree 4, then there are 4(4-2) = 4(2) = 8 points of intersection .

(i.e.,) 8 points of intersection which lie on rectangular hyperbola $x^2-y^2+x+y+1=0$

5.find the asymptodes of $(x-y)(x+y)(x+3y-7)+(2x-3y+1)=0$

Soln.

Given equation is $(x-y)(x+y)(x+3y-7)+(2x-3y+1)=0$

This equation of the form $P_n + P_{n-2}=0$

(i.e.,) $(x-y)(x+y)(x+3y-7)=0$

∴ The asymptotes are $x-y=0$, $x+y=0$, $x+3y-7=0$

Type IV . The equation of the curve n^{th} degree in written in the form of $(ax+by+c)P_{n-1}+F_{n-1}=0$

Where, $P_{n-1},F_{n-1}=0$ denotes the polynomial in x & y of $(n-1)^{th}$ degree then asymptotes of the curve it is given by

$$(ax+by+c)+\lim_{\substack{y\to -\frac{a}{b}x \\ x\to\infty}}\left(\frac{Fn-1}{Pn-1}\right)=0$$

1.find the asymptotes of $x^3+2x^2y-xy^2-2y^3+4y^2+2xy-1=0$

Soln.

Given equation is $x^3+2x^2y-xy^2-2y^3+4y^2+2xy-1=0$

$$X(x^2-y^2)+2y(x^2-y^2)+4y^2+2xy-1=0$$

$(x+2y)(x^2-y^2) + (4y^2+2xy+y-1)=0$

$(x+2y)(x+y)(x-y) + (4y^2+2xy+y-1)=0$

$(x-y)(x+y)(x+2y) + 4y^2+2xy+y-1=0$

There are asymptotes parallel to lines

$x-y=0,\ x+y=0,\ x+2y = 0$

The asymptotes parallel to x-y=0 is given by

$(x-y)(x+y)(x+2y) + 4y^2+2xy+y-1=0$

It is in the form of $(ax+by+c)P_{n-1}+F_{n-1}=0$

Here, $ax+by+c = (x-y)$

$P_{n-1} = (x+y)(x+2y)$

$F_{n-1}=4y^2+2xy+y-1$ &

A=1, b=-1, c=0

The asymptotes are given by,

$$(x-y)+ \lim_{\substack{y\to -\frac{1}{-1}x \\ x\to\infty}} \left(\frac{4y^2+2xy+y-1}{(x+y)(x+2y)}\right) =0$$

$$(x-y)+ \lim_{\substack{\frac{y}{x}\to 1 \\ x\to\infty}} \left(\frac{x^2\left(\frac{4y^2}{x^2}+\frac{2xy}{x^2}+\frac{y}{x^2}-\frac{1}{x^2}\right)}{x^2\left[1+\frac{y}{x}\right]\left[1+\frac{2y}{x}\right]}\right) =0$$

$$(x-y)+ \lim_{x\to\infty} \left(\frac{4(1)+2+\frac{1}{x}-\frac{1}{x^2}}{(1+1)(1+2(1))}\right) =0$$

$$(x-y)+ \lim_{x\to\infty} \left(\frac{6+\frac{1}{x}-\frac{1}{x}}{6}\right) =0$$

$$(x-y)+\left(\frac{6+\frac{1}{\infty}-\frac{1}{\infty}}{6}\right) =0$$

x-y+$\frac{6}{6} = 0$

x-y+1=0

The asymptotes U^r to x+y=0 is given by (x-y)(x+y)(x+2y)+4y^2+2xy+y-1=0

Here , a=1, b=1

(x+y)+ $\lim_{\substack{y\to -\frac{1}{1}x \\ x\to\infty}} \left(\frac{4y^2+2xy+y-1}{(x-y)(x+2y)}\right)$ =0

(x+y)+ $\lim_{\substack{\frac{y}{x}\to -1 \\ x\to\infty}} \left(\frac{x^2\left(\frac{4y^2}{x^2}+\frac{2xy}{x^2}+\frac{y}{x^2}-\frac{1}{x^2}\right)}{x^2\left[1-\frac{y}{x}\right]\left[1+\frac{2y}{x}\right]}\right)$ =0

(x+y)+ $\lim_{x\to\infty} \left(\frac{4(-1)+2(-1)-\frac{1}{x}-\frac{1}{x^2}}{(1+1)(1+2(-1))}\right)$ =0

(x+y)+ $\lim_{x\to\infty} \left(\frac{2-\frac{1}{x}-\frac{1}{x}}{2}\right)$ =0

x+y+$\left(\frac{2}{-2}\right)$=0

x+y-1=0

The asymptotes parallel to x+2y=0 is given by (x-y)(x+y)(x+2y)+4y^2+2xy+y-1=0

Here , a=1, b=2

(x+2y)+ $\lim_{\substack{y\to -\frac{1}{2}x \\ x\to\infty}} \left(\frac{4y^2+2xy+y-1}{(x-y)(x+2y)}\right)$ =0

(x+2y)+ $\lim_{\substack{\frac{y}{x}\to -1/2 \\ x\to\infty}} \left(\frac{x^2\left(\frac{4y^2}{x^2}+\frac{2xy}{x^2}+\frac{y}{x^2}-\frac{1}{x^2}\right)}{x^2\left[1-\frac{y}{x}\right]\left[1+\frac{2y}{x}\right]}\right)$ =0

(x+2y)+ $\lim_{x\to\infty} \left(\frac{4\left(\frac{-1}{2}\right)+2\left(\frac{-1}{2}\right)-\frac{1}{2}x-\frac{1}{x^2}}{\left(1+\frac{1}{2}\right)(1-\frac{1}{2})}\right)$ =0

$$(x+2y)+\lim_{x\to\infty}\left(\frac{1-1+\frac{1}{2x}-\frac{1}{x}}{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}\right)=0$$

$$(x+2y)+\left(\frac{1-1-\frac{1}{2\infty}-\frac{1}{\infty}}{\frac{3}{4}}\right)=0$$

$$x+2y+\left(\frac{0}{\frac{3}{4}}\right)=0$$

$$x+2y+0=0$$

$$x+2y=0$$

CENTRE OF CURVATURE

Let P(x,y) be any point on the curve y=f(x)

Let P be the radius of curvature at p and cut off a distance (pc) equal to p on the normal at p, then the circle with c as centre and cp=p as radius is called the circle of curvature.

The circle of curvature is also called osculating circle and c is called the centre of curvature.

The centre of curvature of the curve y=f(x) at the point x,y is defined by $\bar{x},\bar{y}$

Where, $X=x-y(1+y_1^2)/y_2$

$Y=y+(1+y_1^2)/y_2$

Example:

Find the centre of curvature at the point (c,c) on $xy=c^2$

Solution :

The centre of curvature is defined by x,y

Where , $X=x-y(1+y_1^2)/y_2$

$Y=y+(1+y_1^2)/y_2$

The given curvature is $xy=c^2$

differentiate eqn ① with respect to x

$y_1= -c^2x^{-2}$

$y_1= \frac{-c^2}{x^2}$

$y_1= \frac{-c^2}{c^2}$

$y_1= -1$ at(c,c)

again differentiate ,

$y_2 = -c^2(-2)x^{-3}$

$y_2 =\frac{2c^2}{x^3}$

$y_2 =\frac{2c^2}{c^3}$

$y_2 =\frac{2}{c}$ at (c,c)

$X= c-\frac{(-1)(1+1)}{(2/c)}$

$X= c+\frac{2}{(2/c)}$

$X= c+\frac{2c}{2}$

$X= \frac{2c+2c}{2}$

$X= \frac{4c}{2}$

X=2c

$Y= c+\frac{(1+1)}{(2/c)}$

$Y= c+\frac{2}{(2/c)}$

$Y= c+\frac{2c}{2}$

$Y= \frac{2c+2c}{2}$

$Y= \frac{4c}{2}$

Y=2c

∴ centre of curvature (X,Y) is (2c,2c)

2. find the centre of curvature xy=2 at the points (2,1)

<u>Solution:</u>

The given curvature is xy=2 --------> ①

Differentiate with respect to x in ①

$Y= \frac{2}{x} = 2x^{-1}$

$Y_1 = -2(x)^{-2}$

$Y_1 =\frac{-2}{x^2}$ at point (x,y)=(2,1)

$Y_1 = \frac{-2}{4}$

$Y_1 = \frac{-1}{2}$

Again differentiate with respect to x

$Y_2 = 4(x)^{-3}$

$Y_2 = \frac{4}{x^3}$ at point (x,y)=(2,1)

$Y_2 = \frac{4}{8}$

$Y_2 = \frac{1}{2}$

$X=2- \frac{\left(\frac{-1}{2}\right)\left(1+\frac{1}{4}\right)}{\frac{1}{2}}$

$X=2- \frac{5}{8}(2)$

$X=\frac{2}{1} + \frac{5}{4}$

$X = \frac{8+5}{4}$

$X = \frac{13}{4}$

$Y=1+\frac{\left(1+\left(\frac{-1}{2}\right)^2\right)}{\frac{1}{2}}$

$Y=2-\frac{5}{4}(2)$

$Y=1+\frac{5}{4}(2)$

$Y=\frac{1}{1}+\frac{5}{2}$

$Y=\frac{2+5}{2}$

$Y=\frac{7}{2}$

The centre of curvature (X,Y) is $(\frac{13}{4},\frac{7}{2})$

ENVELOPE

PARAMETER

Consider a family of curve c represented by the equation of (x,y,α) =0 . here α is called a parameter

ONE PARAMETER FAMILY

For different values of α , we get different curves, In such a case, f(x,y, α) =0 is called a one parameter family of a curve

the following equation are example of one parameter family of curve

eg. (i)

Polar coordinates in plane.

Definition

The polar coordinates of a point $P \in R^2$ is the ordered pair (r, θ) defined by the picture.

Theorem (Cartesian-polar transformations)

The Cartesian coordinates of a point P = (r, θ) in the first quadrant are given by

$x = r\cos(\theta)$, $y = r\sin(\theta)$.

The polar coordinates of a point P = (x, y) in the first quadrant are given by

$r = \sqrt{x^2 + y^2}$, $\theta = \arctan(\frac{y}{x})$

Change of order of integration

Change of order of integration is done to make the evaluation of integral easierThe following are very important when the change of order of integration takes place

1. If the limits of the inner integral is a function of x (or function of y) then the firstintegration should be with respect to y (or with respect to x)
2. Draw the region of integration by using the given limits
3. If the integration is first with respect to x keeping y as a constant then consider thehorizontal strip and find the new limits accordingly
4. If the integration is first with respect to y keeping x a constant then consider thevertical strip and find the new limits accordingly
5. After find the new limits evaluate the inner integral first and then the outer integral.